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ON THE TWIST OF A TORTUOUS CURVE

By Paul Saurel

To explain what we shall understand by the expression twist of a tortuous curve, let us suppose that at any given point of a curve the tangent, the principal normal, and the binormal have been drawn. We shall imagine that a moving point describes the curve and we shall take as the positive direction of the tangent the direction of the motion of the tracing point as it passes through the given point. As the tracing point continues its motion it will detach itself from the rectifying plane; we shall take as the positive end of the principal normal the end which lies on the same side of the rectifying plane as the moving point in its new position. Finally, we shall choose the positive direction of the binormal so that the tangent, the principal normal, and the binormal shall form a set of rectangular axes congruent with the axes of reference.

As the moving point passes through the given point it will detach itself from the osculating plane. If its new position be on the same side of the osculating plane as the positive end of the binormal, we shall say that the twist of the curve at the given point and for the given motion of the tracing point is positive; if, on the contrary, the new position of the tracing point be on the same side of the osculating plane as the negative end of the binormal, we shall say that the twist of the curve at the given point and for the given motion of the tracing point is negative.

If we suppose that the coordinates X, Y, Z of a point on the curve are given by series of integral powers of a parameter t, convergent within a certain interval, we may write

$$(1)$$

$$X = x + x't + \frac{x''}{2} t^2 + \cdots, \quad Y = y + y't + \frac{y''}{2} t^2 + \cdots, \quad Z = z + z't + \frac{z''}{2} t^2 + \cdots,$$

in which x, y, z denote the coordinates of the point which corresponds to the value zero of the parameter and x', y', z', x'', y'', z'', $\cdots$ are the successive derivatives of the coordinates for that point. In the following discussion we shall suppose that the tracing point moves in the direction in which t increases.

The tangent at the point x, y, z is defined as the limiting position of the line which joins x, y, z with a neighboring point of the curve. From this it follows that the equations of the tangent are

$$\frac{X - x}{x^p} = \frac{Y - y}{y^p} = \frac{Z - z}{z^p}, \tag{2}$$

in which x^p, y^p, z^p denote the first set of derivatives which is not identically equal to zero. The direction cosines of the positive end of the tangent are equal to these three derivatives multiplied by the same positive number.

The osculating plane of the curve at the point x, y, z is defined as the limiting position of the plane which passes through the tangent at x, y, z and a neighboring point of the curve. From this it follows that the equation of the osculating plane is

$$\begin{vmatrix} X - x & Y - y & Z - z \\ x^p & y^p & z^p \\ x^q & y^q & z^q \end{vmatrix} = 0, \tag{3}$$

in which $q > p$ and x^q, y^q, z^q denote the first set of derivatives which are not proportional to x^p, y^p, z^p. We take as the positive end of the binormal the end whose direction cosines are equal to

$$y^p z^q - z^p y^q, \; z^p x^q - x^p z^q, \; x^p y^q - y^p x^q, \tag{4}$$

multiplied by the same positive number.

Since the tangent, the principal normal, and the binormal are to form a set of axes congruent with the axes of reference, the direction cosines of the positive end of the principal normal must be equal to the determinants

$$\begin{vmatrix} z^p x^q - x^p z^q & x^p y^q - y^p x^q \\ y^p & z^p \end{vmatrix}, \; \begin{vmatrix} x^p y^q - y^p x^q & y^p z^q - z^p y^q \\ z^p & x^p \end{vmatrix},$$

$$\begin{vmatrix} y^p z^q - z^p y^q & z^p x^q - x^p z^q \\ x^p & y^p \end{vmatrix}, \tag{5}$$

multiplied by the same positive number. The equation of the rectifying plane will accordingly be:

$$\begin{vmatrix} X - x & Y - y & Z - z \\ y^p z^q - z^p y^q & z^p x^q - x^p z^q & x^p y^q - y^p x^q \\ x^p & y^p & z^p \end{vmatrix} = 0. \tag{6}$$

From this equation it follows without difficulty that the distance from the

rectifying plane to a point which corresponds to a positive value of t and which lies in the immediate neighborhood of x, y, z is approximately equal to the determinant

$$\begin{vmatrix} x^q & y^q & z^q \\ y^p z^q - z^p y^q & z^p x^q - x^p z^q & x^p y^q - y^p x^q \\ x^p & y^p & z^p \end{vmatrix} \qquad (7)$$

multiplied by a positive number. Expanding this determinant in terms of the elements of the second row we find that it is equal to the sum of the squares of the expressions 4; it follows that the new position of the moving point is on the same side of the rectifying plane as the positive end of the principal normal.

In like manner, it follows from equation 3 that the distance from the osculating plane to a point in the immediate neighborhood of x, y, z is approximately equal to the product of a positive number by the determinant

$$\begin{vmatrix} x^p & y^p & z^p \\ x^q & y^q & z^q \\ x^r & y^r & z^r \end{vmatrix}, \qquad (8)$$

in which $r > q$ and x^r, y^r, z^r are the first set of derivatives for which the above determinant does not reduce to zero. The new position of the tracing point will therefore lie on the positive or on the negative side of the osculating plane according as the determinant 8 is positive or negative.

We have thus established the theorem: If the curve 1 is described by a point which moves in the direction of increasing values of t, the twist of the curve at the point x, y, z will be positive or negative according as the determinant 8 is positive or negative.

To apply the foregoing theory to the determination of the twist when the tracing point moves in the direction of decreasing values of t we must replace t in equations 1 by $-t'$. Every derivative of odd order with respect to t is equal to the negative of the derivative of the same order with respect to t'. It then follows at once from 8 that when we pass through a given point in opposite directions the twist does or does not change sign according as $p + q + r$ is odd or even. At an ordinary point of the curve $p = 1$, $q = 2$, $r = 3$; the twist, accordingly, does not change sign when the tracing point describes the curve in the opposite direction.

We can go further and determine the character of the twist at a singular

point of the curve. But before doing this it is necessary to state the analytical characteristics of each of von Staudt's eight types of singular points. As is well known, von Staudt's classification is based upon the behavior of the tracing point and the accompanying tangent line and osculating plane. At a given point each of these elements, point, line, and plane, may either continue its motion in the same direction or it may stop and begin to move in the opposite direction. In a previous note[*] we have shown that if the tangent at a given point of a curve be taken as x axis, the principal normal as y axis, and the binormal as z axis, the equations of the curve take the form

$$X = at^{p'} + a_1 t^{p'+1} + \cdots, \quad Y = bt^{q'} + b_1 t^{q'+1} + \cdots, \quad Z = ct^{r'} + c_1 t^{r'+1} + \cdots,$$
$$p' < q' < r', \qquad (9)$$

and moreover that, at the given point, the motion of the tracing point persists or reverses according as p' is odd or even, that the motion of the tangent line persists or reverses according as $p' + q'$ is odd or even, and that the motion of the osculating plane persists or reverses according as $q' + r'$ is odd or even.

From these criteria it is not hard to deduce the criteria which apply when the curve is given by equations of the form 1. It can be shown that the motion of the tracing point persists or reverses according as the index p of the first set of non-vanishing derivatives x^p, y^p, z^p is odd or even, that the motion of the tangent line persists or reverses according as the sum $p + q$ of the indices in the first set of non-vanishing expressions of the form 4 is odd or even, and finally that the motion of the osculating plane persists or reverses according as the sum of the indices $q + r$ in the first non-vanishing determinant of the form 8 is odd or even.[†]

To establish this proposition we take the equations of the curve in the form 9 and transform them to rectangular axes passing through the given point and parallel to the axes to which equations 1 are referred. If the direction cosines of the tangent, the principal normal, and the binormal with reference to these new axes be respectively (l_1, m_1, n_1), (l_2, m_2, n_2), and (l_3, m_3, n_3), the equations of the curve referred to the new axes will be

$$X = l_1(at^{p'} + a_1 t^{p'+1} + \cdots) + l_2(bt^{q'} + b_1 t^{q'+1} + \cdots) + l_3(ct^{r'} + c_1 t^{r'+1} + \cdots),$$
$$Y = m_1(at^{p'} + a_1 t^{p'+1} + \cdots) + m_2(bt^{q'} + b_1 t^{q'+1} + \cdots) + m_3(ct^{r'} + c_1 t^{r'+1} + \cdots), \quad (10)$$
$$Z = n_1(at^{p'} + a_1 t^{p'+1} + \cdots) + n_2(bt^{q'} + b_1 t^{q'+1} + \cdots) + n_3(ct^{r'} + c_1 t^{r'+1} + \cdots).$$

[*] See p. 3, above.

[†] H. B. Fine, *American Journal of Mathematics*, vol. 8, p. 156; 1886.

By applying to these equations the theory developed in connection with equations 1 it will be found without difficulty that the first set of non-vanishing derivatives are of order p', that the indices in the first set of non-vanishing expressions of the form 4 are p' and q', and that the indices in the first non-vanishing determinant of the form 8 are p', q', and r'. To establish this it is necessary to remember that

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 1$$

and that any element of this determinant is equal to its minor.

If equations 1 and 9 refer to the same curve it follows at once that the indices p, q, and r which appear in 2, 4, and 8 are respectively equal to the indices p', q', and r' which appear in 9. Thus the numbers which determine the type to which the singular point under consideration belongs are the same as the numbers which determine the character of the twist at that point. If the characteristics of each of the types of singular points be written it will be found that for four of the eight types $p + q + r$ is even, while for the other four types this sum is odd. It follows that for four of the eight classes the twist of the curve at the given point is independent of the direction in which the tracing point passes through it, while for the remaining four classes the twist changes sign when the direction in which the tracing point describes the curve is reversed.

In conclusion, it should be stated that the twist of a curve at a singular point has been determined by Staude* for each of the following sets of values of p, q, r: (1, 2, 3), (1, 2, 4), (1, 3, 4), (1, 3, 5), (2, 3, 4), (2, 3, 5), (2, 4, 5), (2, 4, 6). These eight sets of values furnish the simplest examples of the eight types of singular points.

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*American Journal of Mathematics, vol. 17, p. 359; 1895.